

First-Order Logic

Part One

Recap from Last Time

Recap So Far

- A ***propositional variable*** is a variable that is either true or false.
- The ***propositional connectives*** are as follows:
 - Negation (“not”): $\neg p$
 - Conjunction (“and”): $p \wedge q$
 - Disjunction (“or”): $p \vee q$
 - Truth (“true”): \top
 - Falsity (“false”): \perp
 - Implication (“implies”): $p \rightarrow q$
 - Biconditional (“if and only if”): $p \leftrightarrow q$

p	q	$p \rightarrow q$
F	F	—
F	T	—
T	F	—
T	T	—

What's the truth table for the \rightarrow connective?

Answer at

<https://pollev.com/cs103aut23>

What's the negation of $p \rightarrow q$?

Answer at

<https://pollev.com/cs103aut23>

Operator Precedence

- How do we parse this statement?

$$\neg x \rightarrow y \vee z \rightarrow x \vee y \wedge z$$

- Operator precedence for propositional logic:

\neg

\wedge

\vee

\rightarrow

\leftrightarrow

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- The main points to remember:
 - \neg binds to whatever immediately follows it.
 - \wedge and \vee bind more tightly than \rightarrow .
- We will commonly write expressions like $p \wedge q \rightarrow r$ without adding parentheses.
- For more complex expressions, we'll try to add parentheses.
- Confused? ***Please ask!***

Logic at the US Supreme Court

- There's a current case before the US Supreme Court concerning whether a piece of text in a law should be interpreted as

$$\neg(A \wedge B \wedge C)$$

or as

$$\neg A \wedge \neg B \wedge \neg C.$$

- You can *listen to the oral arguments* on the Supreme Court's website. They regularly throw around terms like "conjunction," "disjunction," and "distributive."

New Stuff!

First-Order Logic

What is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
 - ***predicates*** that describe properties of objects,
 - ***functions*** that map objects to one another, and
 - ***quantifiers*** that allow us to reason about multiple objects.

Some Examples

Likes(You, Eggs) \wedge Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)

Learns(You, History) \vee ForeverRepeats(You, History)

In(MyHeart, Havana) \wedge TookBackTo(Him, Me, EastAtlanta)

Likes(You, Eggs) ∧ Likes(You, Tomato) → Likes(You, Shakshuka)

Learns(You, History) ∨ ForeverRepeats(You, History)

In(MyHeart, Havana) ∧ TookBackTo(Him, Me, EastAtlanta)

These blue terms are called *constant symbols*. Unlike propositional variables, they refer to objects, not propositions.

Likes(You, Eggs) \wedge Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)

Learns(You, History) \vee ForeverRepeats(You, History)

In(MyHeart, Havana) \wedge TookBackTo(Him, Me, EastAtlanta)

The red things that look like function calls are called *predicates*. Predicates take objects as arguments and evaluate to true or false.

Likes(You, Eggs) \wedge Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)

Learns(You, History) \vee ForeverRepeats(You, History)

In(MyHeart, Havana) \wedge TookBackTo(Him, Me, EastAtlanta)

What remains are traditional propositional connectives. Because each predicate evaluates to true or false, we can connect the truth values of predicates using normal propositional connectives.

Reasoning about Objects

- To reason about objects, first-order logic uses ***predicates***.
- Examples:

Cute(Quokka)

ArgueIncessantly(Democrats, Republicans)

- Applying a predicate to arguments produces a proposition, which is either true or false.
- Typically, when you're working in FOL, you'll have a list of predicates, what they stand for, and how many arguments they take. It'll be given separately than the formulas you write.

First-Order Formulas

- Formulas in first-order logic can be constructed from predicates applied to objects:

$Cute(a) \rightarrow Dikdik(a) \vee Kitty(a) \vee Puppy(a)$

$Succeeds(You) \leftrightarrow Practices(You)$

$x < 8 \rightarrow x < 137$

The less-than sign is just another predicate. Binary predicates are sometimes written in *infix notation* this way.

Numbers are not "built in" to first-order logic. They're constant symbols just like "You" and "a" above.

Equality

- First-order logic is equipped with a special predicate $=$ that says whether two objects are equal to one another.
- Equality is a part of first-order logic, just as \rightarrow and \neg are.
- Examples:

TomMarvoloRiddle = LordVoldemort

MorningStar = EveningStar

- Equality can only be applied to **objects**; to state that two **propositions** are equal, use \leftrightarrow .

Let's see some more examples.

FavoriteMovieOf(You) ≠ FavoriteMovieOf(Date) ∧
StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))

These purple terms are *functions*. Functions take objects as input and produce objects as output.

*FavoriteMovieOf(You) ≠ FavoriteMovieOf(Date) ∧
StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))*

Functions

- First-order logic allows **functions** that return objects associated with other objects.
- Examples:

ColorOf(Money)

MedianOf(x, y, z)

$x + y$

- As with predicates, functions can take in any number of arguments, but always return a single value.
- Functions evaluate to **objects**, not **propositions**.

Objects and Propositions

- When working in first-order logic, be careful to keep objects (actual things) and propositions (true or false) separate.

- You cannot apply connectives to objects:

Venus \rightarrow *TheSun*

- You cannot apply functions to propositions:

StarOf(IsRed(Sun) \wedge IsGreen(Mars))

- Ever get confused? *Just ask!*

The Type-Checking Table

	... operate on and produce
Connectives (\leftrightarrow , \wedge , etc.) ...	propositions	a proposition
Predicates ($=$, etc.) ...	objects	a proposition
Functions ...	objects	an object

One last (and major) change

Some muggle is intelligent.

$\exists m. (Muggle(m) \wedge Intelligent(m))$

\exists is the **existential quantifier** and says "for some choice of m , the following is true."

The Existential Quantifier

- A statement of the form

$\exists x.$ *some-formula*

is true if there exists a choice of x where ***some-formula*** is true when that x is plugged into it.

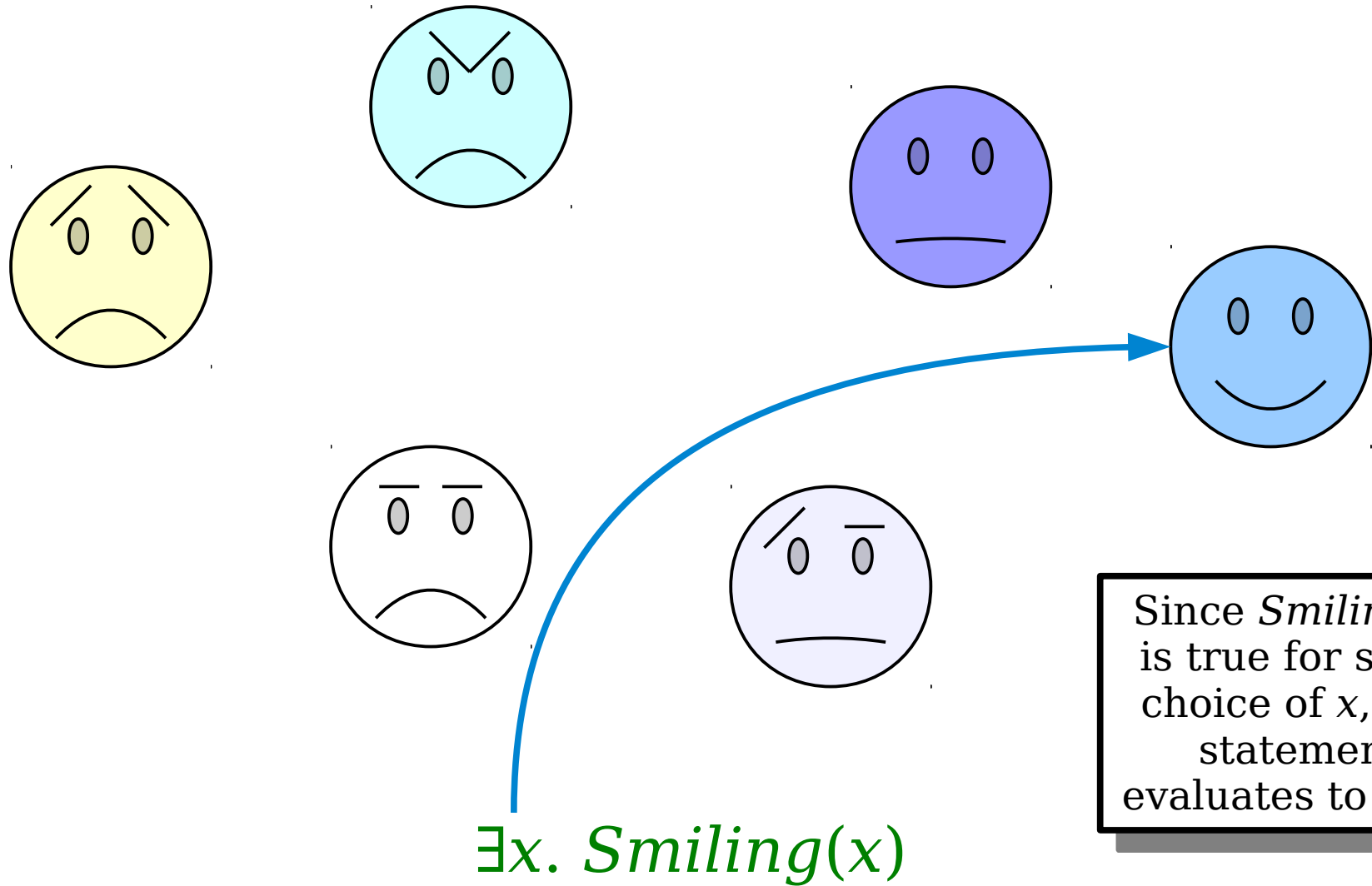
- Examples:

$\exists x. (Even(x) \wedge Prime(x))$

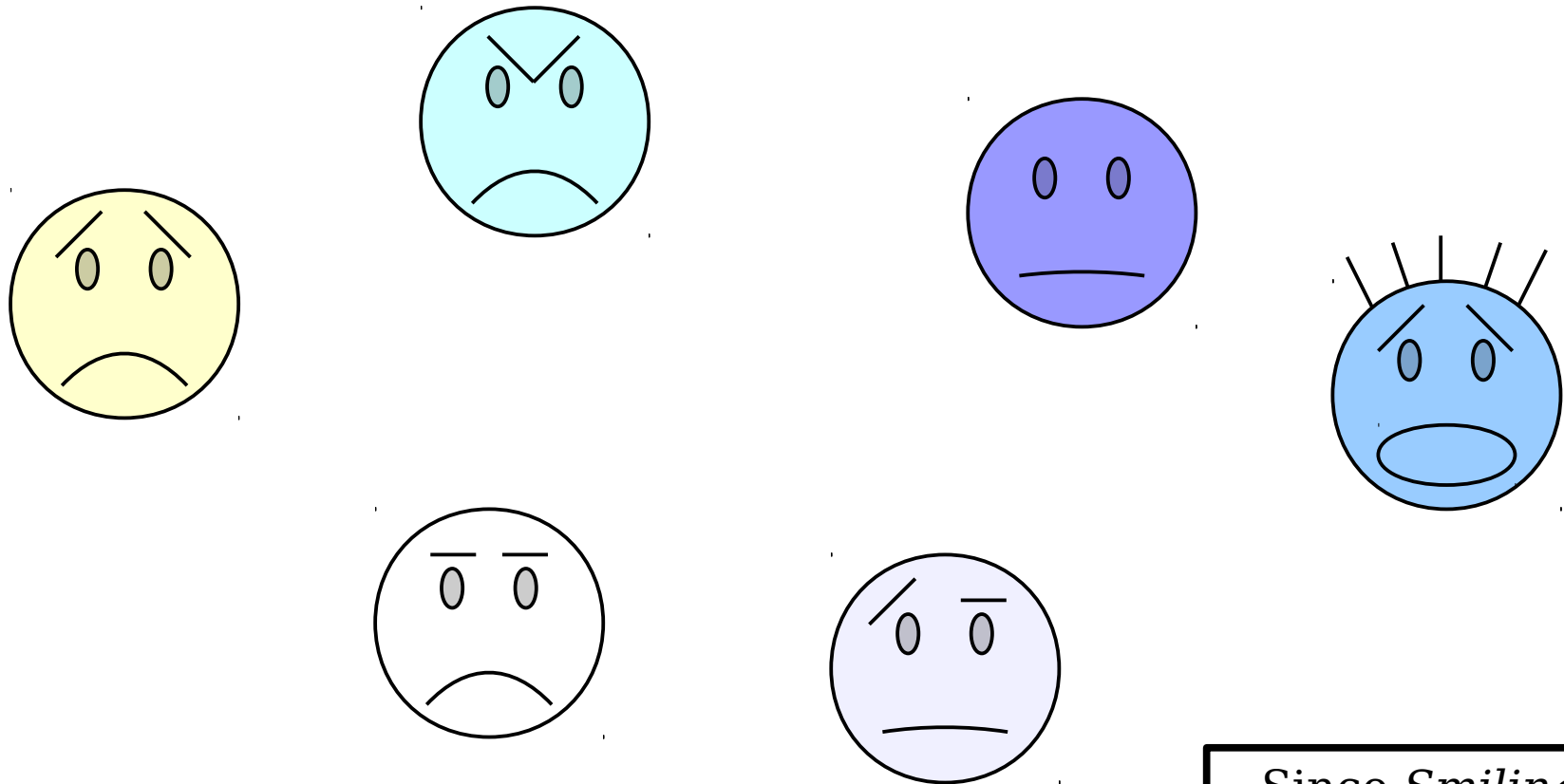
$\exists x. (TallerThan(x, me) \wedge WeighsLessThan(x, me))$

$(\exists w. Will(w)) \rightarrow (\exists x. Way(x))$

The Existential Quantifier



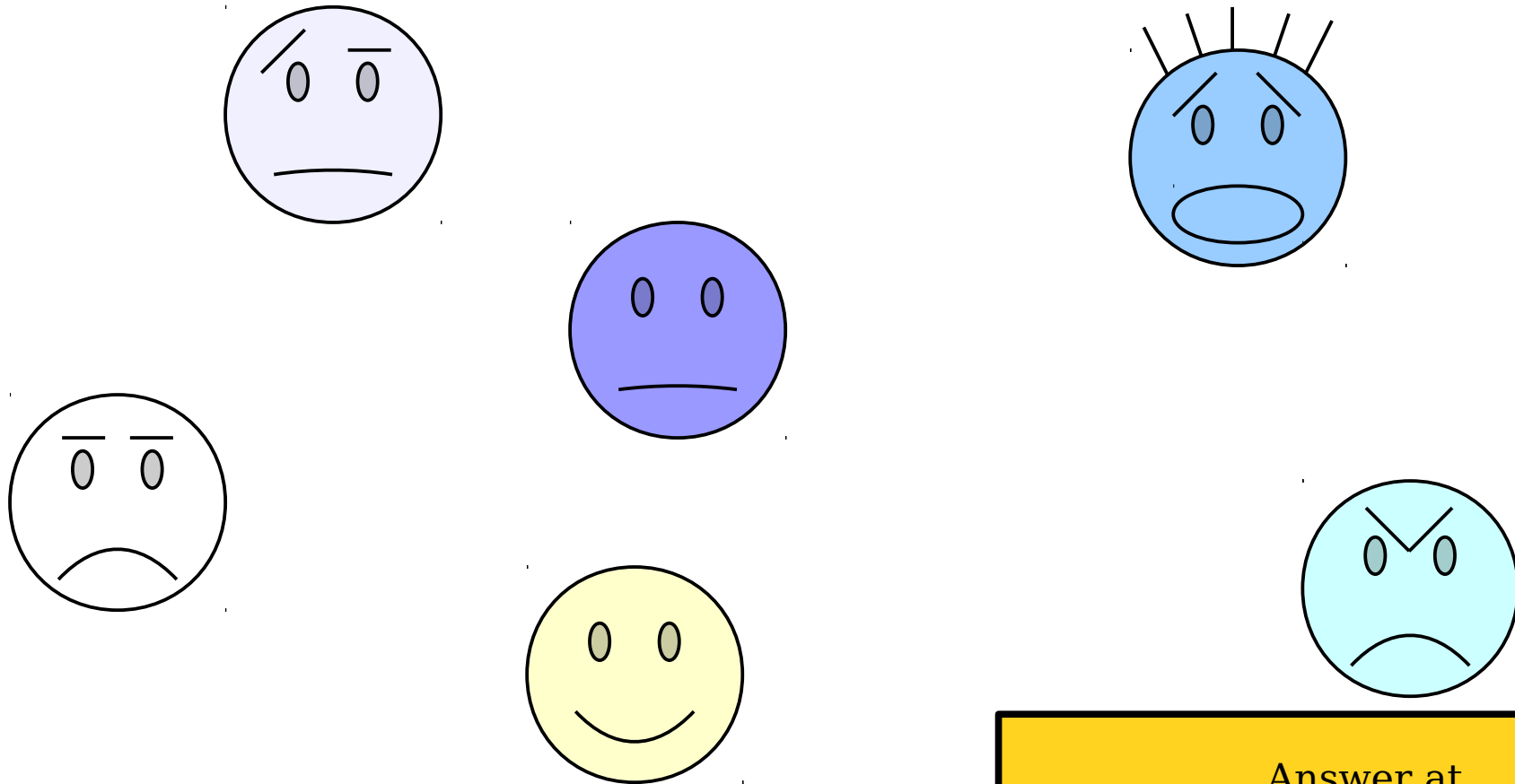
The Existential Quantifier



~~$\exists x. Smiling(x)$~~

Since $Smiling(x)$ is not true for any choice of x , this statement evaluates to false.

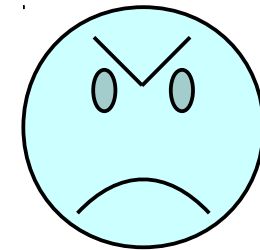
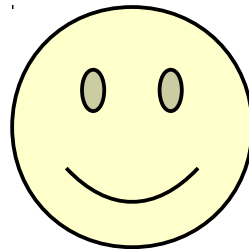
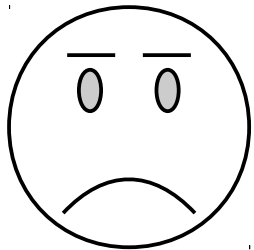
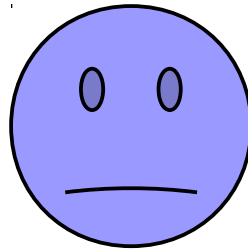
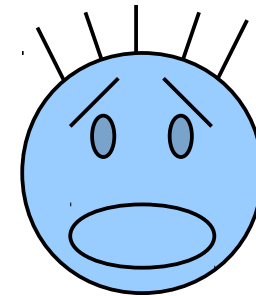
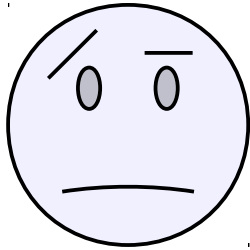
The Existential Quantifier



Answer at
<https://pollev.com/cs103aut23>

$$(\exists x. \textit{Smiling}(x)) \rightarrow (\exists y. \textit{WearingHat}(y))$$

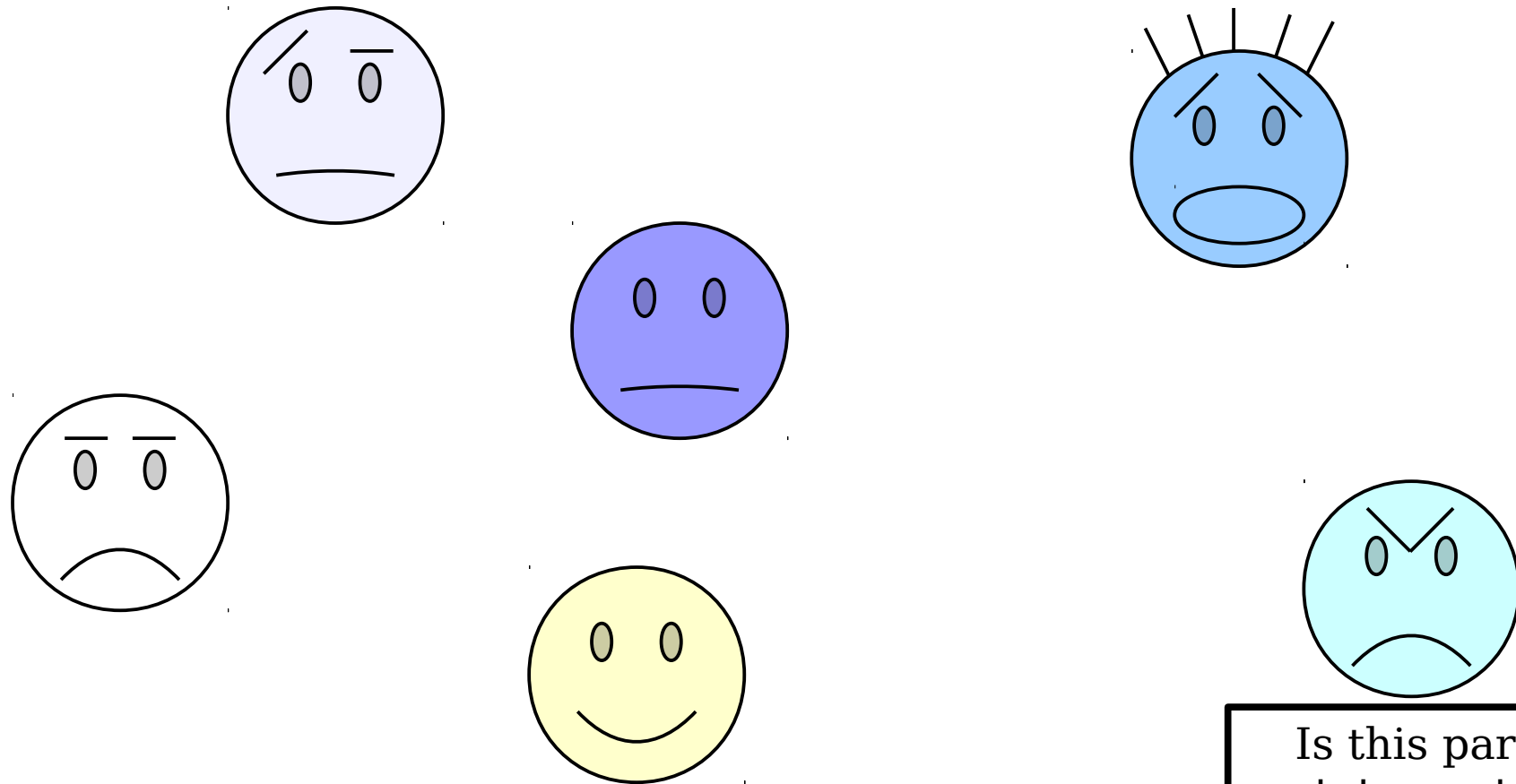
The Existential Quantifier



Is this part of the statement true or false?

$(\exists x. \textit{Smiling}(x)) \rightarrow (\exists y. \textit{WearingHat}(y))$

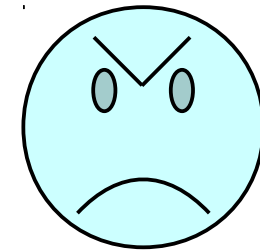
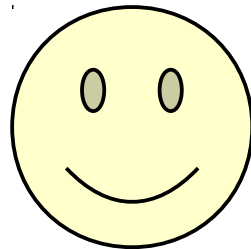
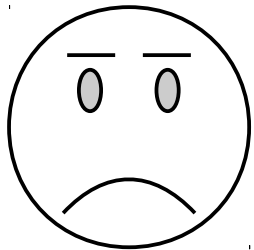
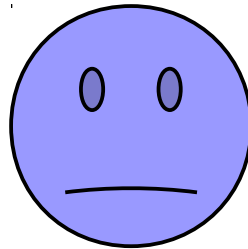
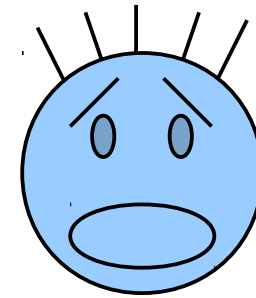
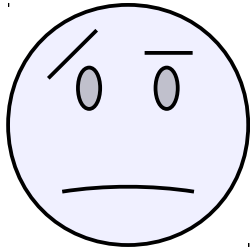
The Existential Quantifier



Is this part of the statement true or false?

$(\exists x. \textit{Smiling}(x)) \rightarrow (\exists y. \textit{WearingHat}(y))$

The Existential Quantifier



Is this overall
statement true or
false?

~~$(\exists x. Smiling(x)) \rightarrow (\exists y. WearingHat(y))$~~

Fun with Edge Cases

Existentially-quantified statements are false in an empty world, since nothing exists, period!

~~$\exists x. \textit{Smiling}(x)$~~

Some Technical Details

Variables and Quantifiers

- Each quantifier has two parts:
 - the variable that is introduced, and
 - the statement that's being quantified.
- The variable introduced is scoped just to the statement being quantified.

$(\exists x. \text{Loves}(\text{You}, x)) \wedge (\exists y. \text{Loves}(y, \text{You}))$

The variable x
just lives here.

The variable y
just lives here.

Variables and Quantifiers

- Each quantifier has two parts:
 - the variable that is introduced, and
 - the statement that's being quantified.
- The variable introduced is scoped just to the statement being quantified.

$$(\exists x. \text{Loves}(\text{You}, x)) \wedge (\exists x. \text{Loves}(x, \text{You}))$$

The variable x
just lives here.

A different variable,
also named x , just
lives here.

Operator Precedence (Again)

- When writing out a formula in first-order logic, quantifiers have precedence just below \neg .
- The statement

$$\exists x. P(x) \wedge R(x) \wedge Q(x)$$

is parsed like this:

$$(\exists x. P(x)) \wedge (R(x) \wedge Q(x))$$

- This is syntactically invalid because the variable x is out of scope in the back half of the formula.
- To ensure that x is properly quantified, explicitly put parentheses around the region you want to quantify:

$$\exists x. (P(x) \wedge R(x) \wedge Q(x))$$

“For any natural number n ,
 n is even if and only if n^2 is even”

$\forall n. (n \in \mathbb{N} \rightarrow (Even(n) \leftrightarrow Even(n^2)))$

\forall is the **universal quantifier**
and says “for any choice of n ,
the following is true.”

The Universal Quantifier

- A statement of the form

$\forall x.$ *some-formula*

is true if, for every choice of x , the statement ***some-formula*** is true when x is plugged into it.

- Examples:

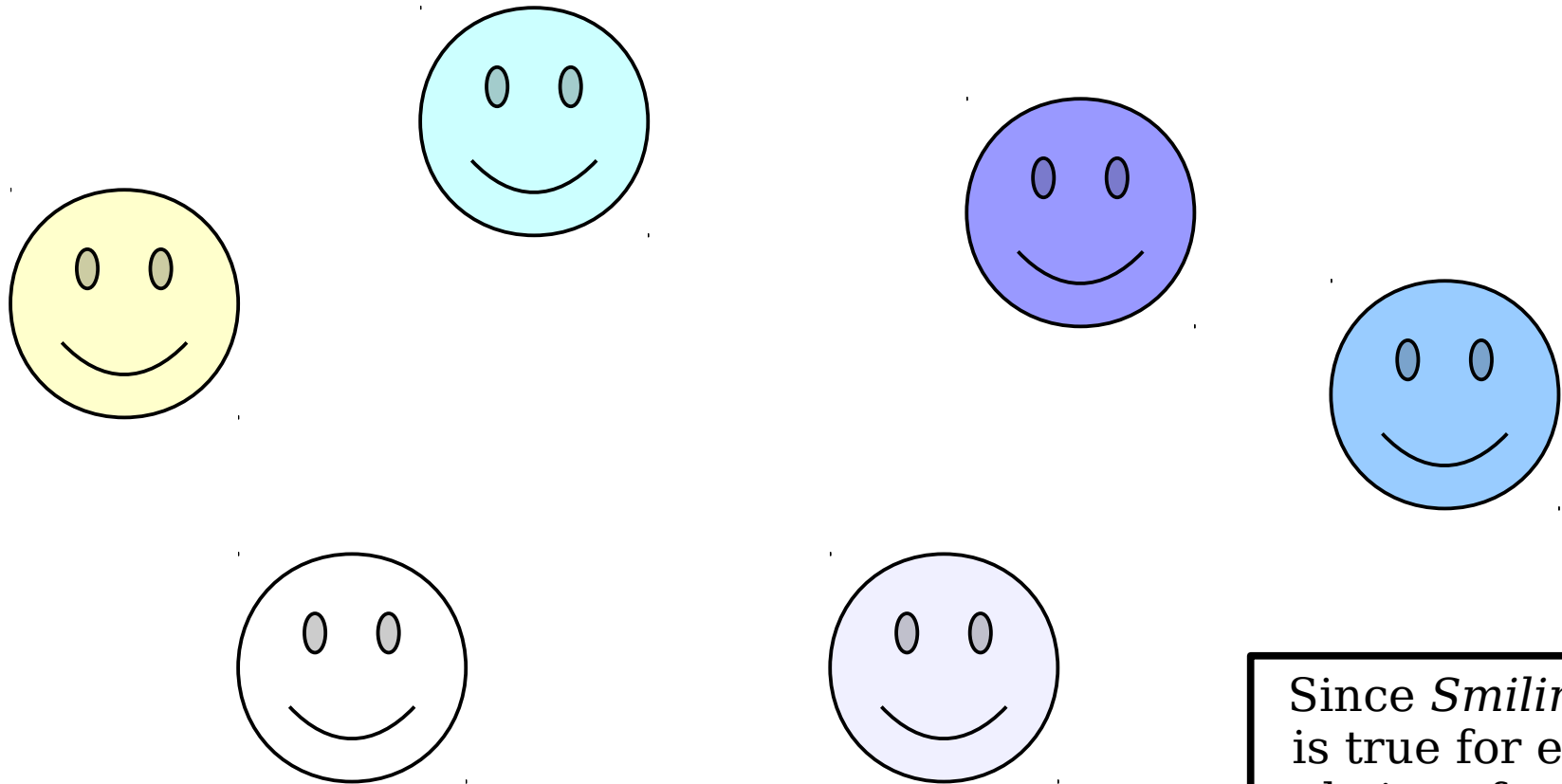
$\forall p. (Puppy(p) \rightarrow Cute(p))$

$\forall a. (EatsPlants(a) \vee EatsAnimals(a))$

$Tallest(SultanKösen) \rightarrow$

$\forall x. (SultanKösen \neq x \rightarrow ShorterThan(x, SultanKösen))$

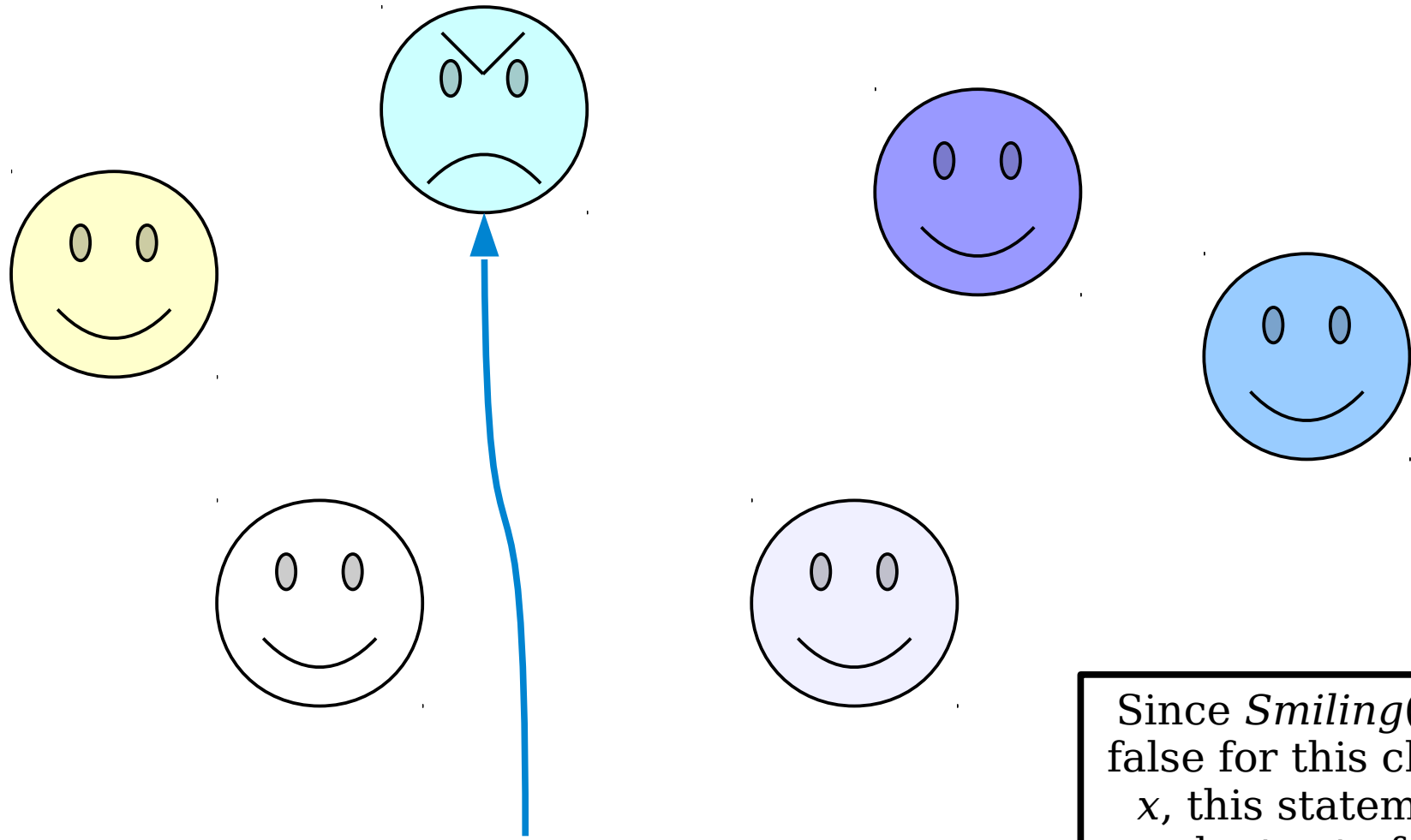
The Universal Quantifier



$\forall x. Smiling(x)$

Since *Smiling*(*x*)
is true for every
choice of *x*, this
statement
evaluates to true.

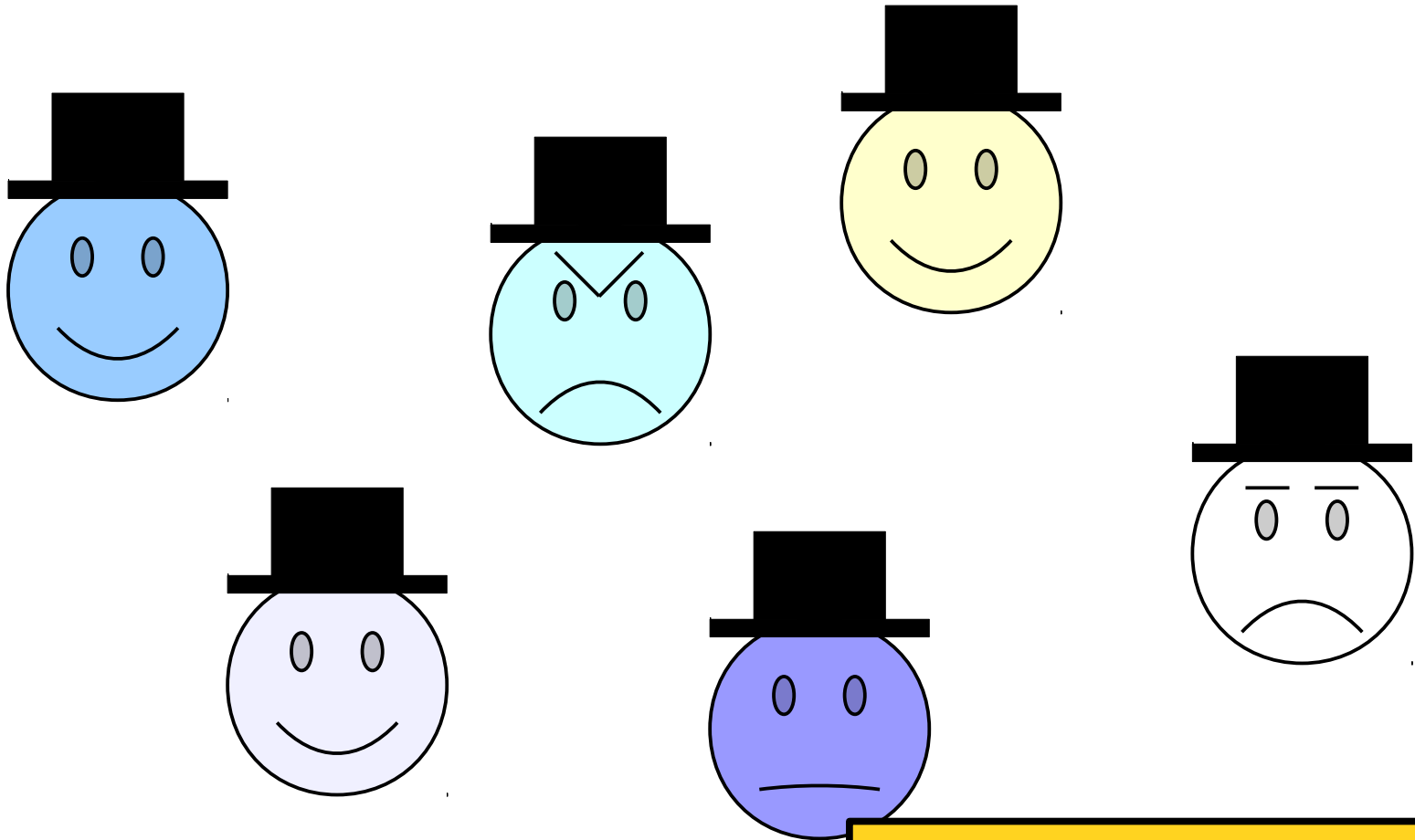
The Universal Quantifier



~~$\forall x. Smiling(x)$~~

Since $Smiling(x)$ is false for this choice x , this statement evaluates to false.

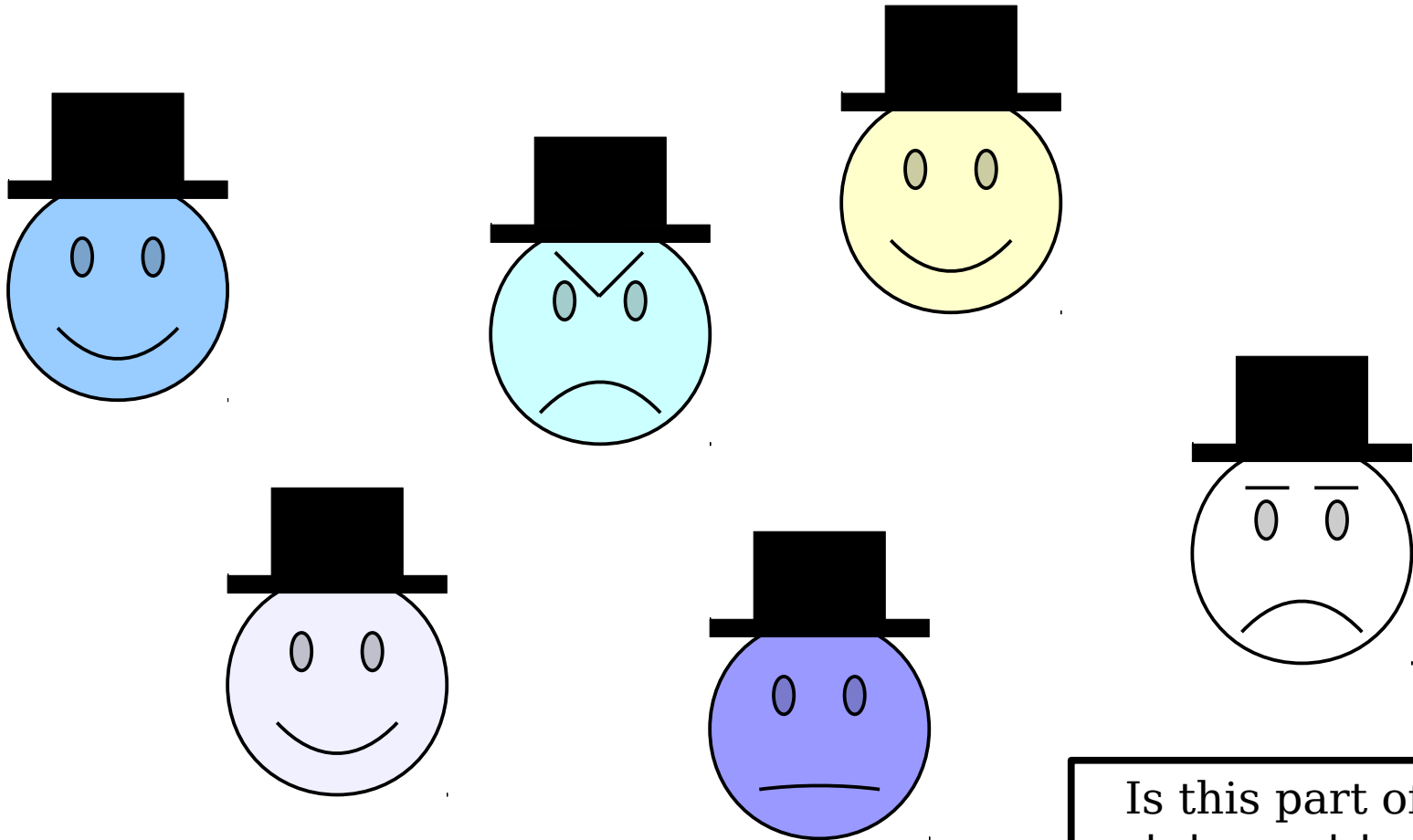
The Universal Quantifier



Answer at
<https://pollev.com/cs103aut23>

$(\forall x. \textit{Smiling}(x)) \rightarrow (\forall y. \textit{WearingHat}(y))$

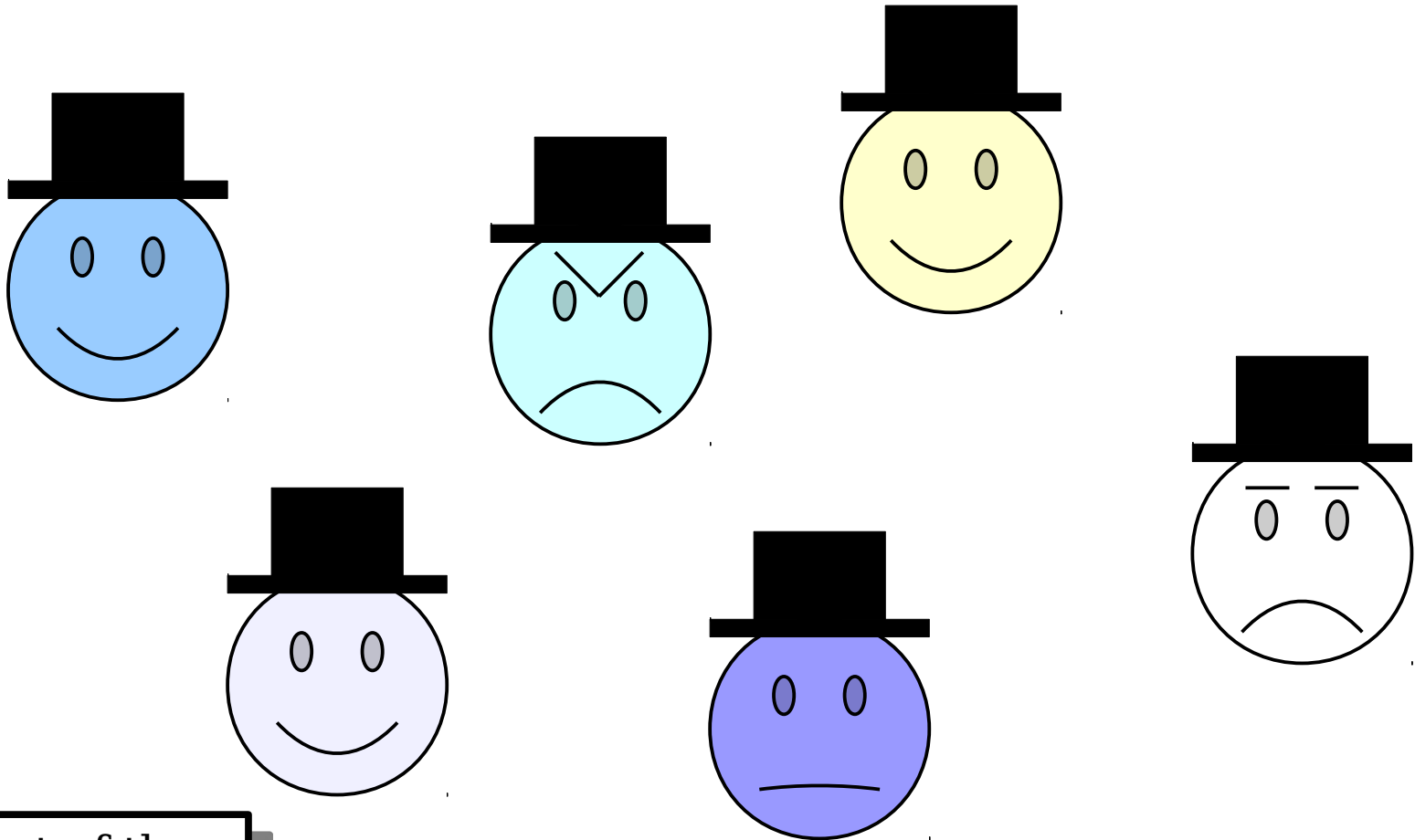
The Universal Quantifier



Is this part of the statement true or false?

$(\forall x. \textit{Smiling}(x)) \rightarrow (\forall y. \textit{WearingHat}(y))$

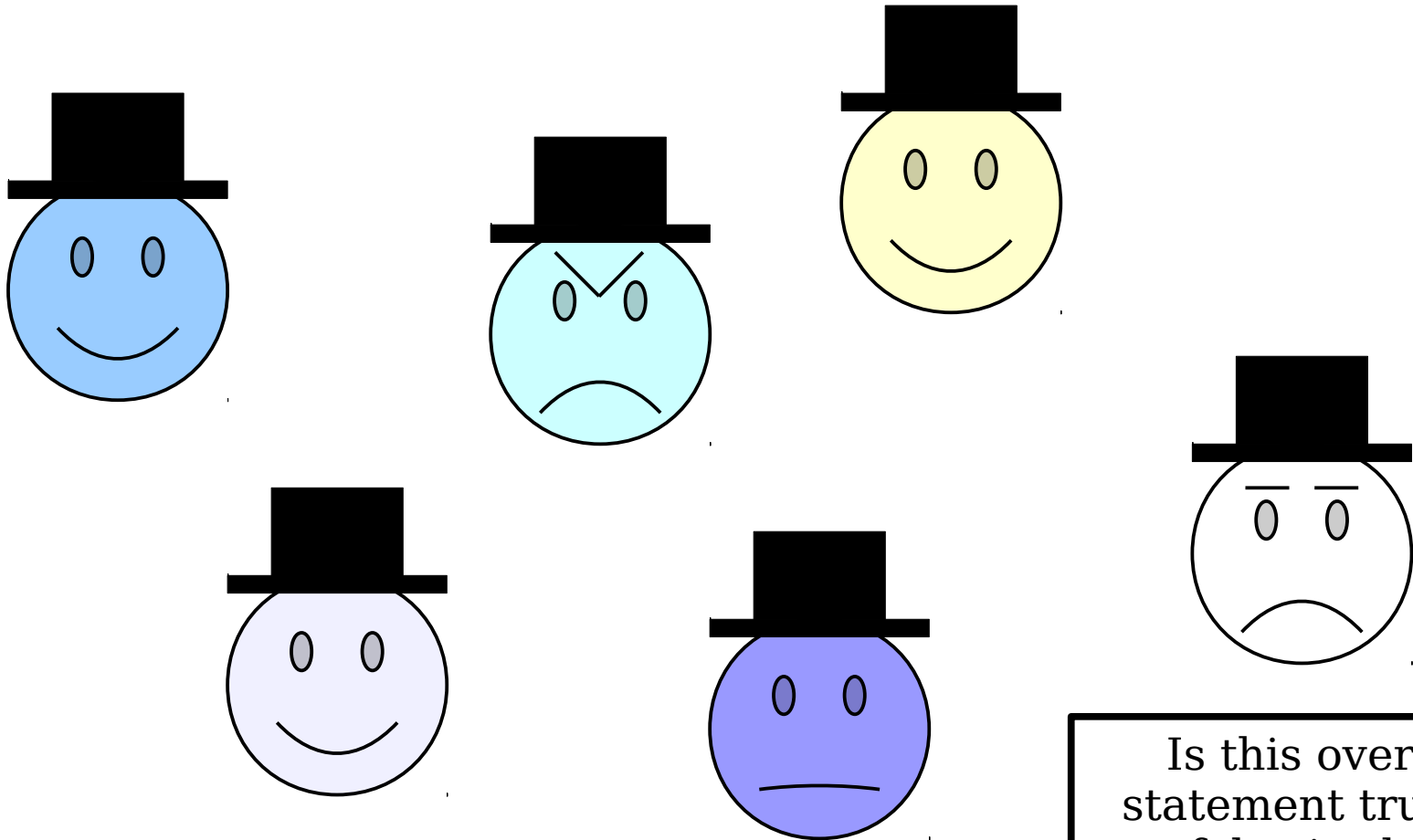
The Universal Quantifier



Is this part of the statement true or false?

~~$(\forall x. \textit{Smiling}(x))$~~ $\rightarrow (\forall y. \textit{WearingHat}(y))$

The Universal Quantifier



Is this overall statement true or false in this scenario?

$(\forall x. \textit{Smiling}(x)) \rightarrow (\forall y. \textit{WearingHat}(y))$

Fun with Edge Cases

Universally-quantified statements are said to be *vacuously true* in empty worlds.

$\forall x. \textit{Smiling}(x)$

Translating into First-Order Logic

Translating Into Logic

- First-order logic is an excellent tool for manipulating definitions and theorems to learn more about them.
- Need to take a negation? Translate your statement into FOL, negate it, then translate it back.
- Want to prove something by contrapositive? Translate your implication into FOL, take the contrapositive, then translate it back.

Translating Into Logic

- When translating from English into first-order logic, we recommend that you ***think of first-order logic as a mathematical programming language.***
- Your goal is to learn how to combine basic concepts (quantifiers, connectives, etc.) together in ways that say what you mean.

Using the predicates

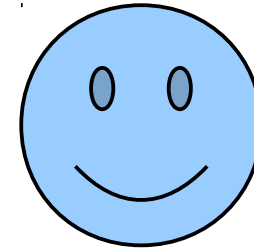
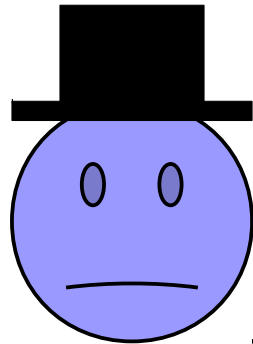
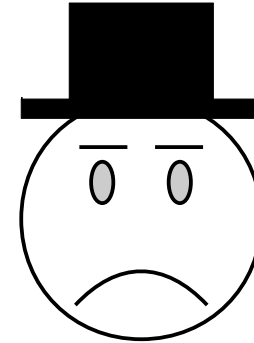
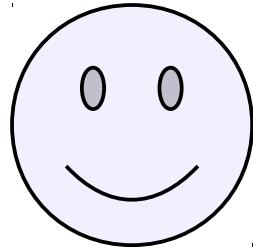
- *Smiling*(x), which states that x is smiling, and
- *WearingHat*(x), which states that x is wearing a hat,

write a sentence in first-order logic that says

some smiling person wears a hat.

Answer at

<https://pollev.com/cs103aut23>



“Some smiling person wears a hat.” ***False***

$\exists x. (Smiling(x) \wedge WearingHat(x))$ ***False***

~~$\exists x. (Smiling(x) \rightarrow WearingHat(x))$~~ ***True***

“Some P is a Q ”

translates as

$\exists x. (P(x) \wedge Q(x))$

Useful Intuition:

Existentially-quantified statements are false unless there's a positive example.

$$\exists x. (P(x) \wedge Q(x))$$

If x is an example, it must have property P on top of property Q .

Using the predicates

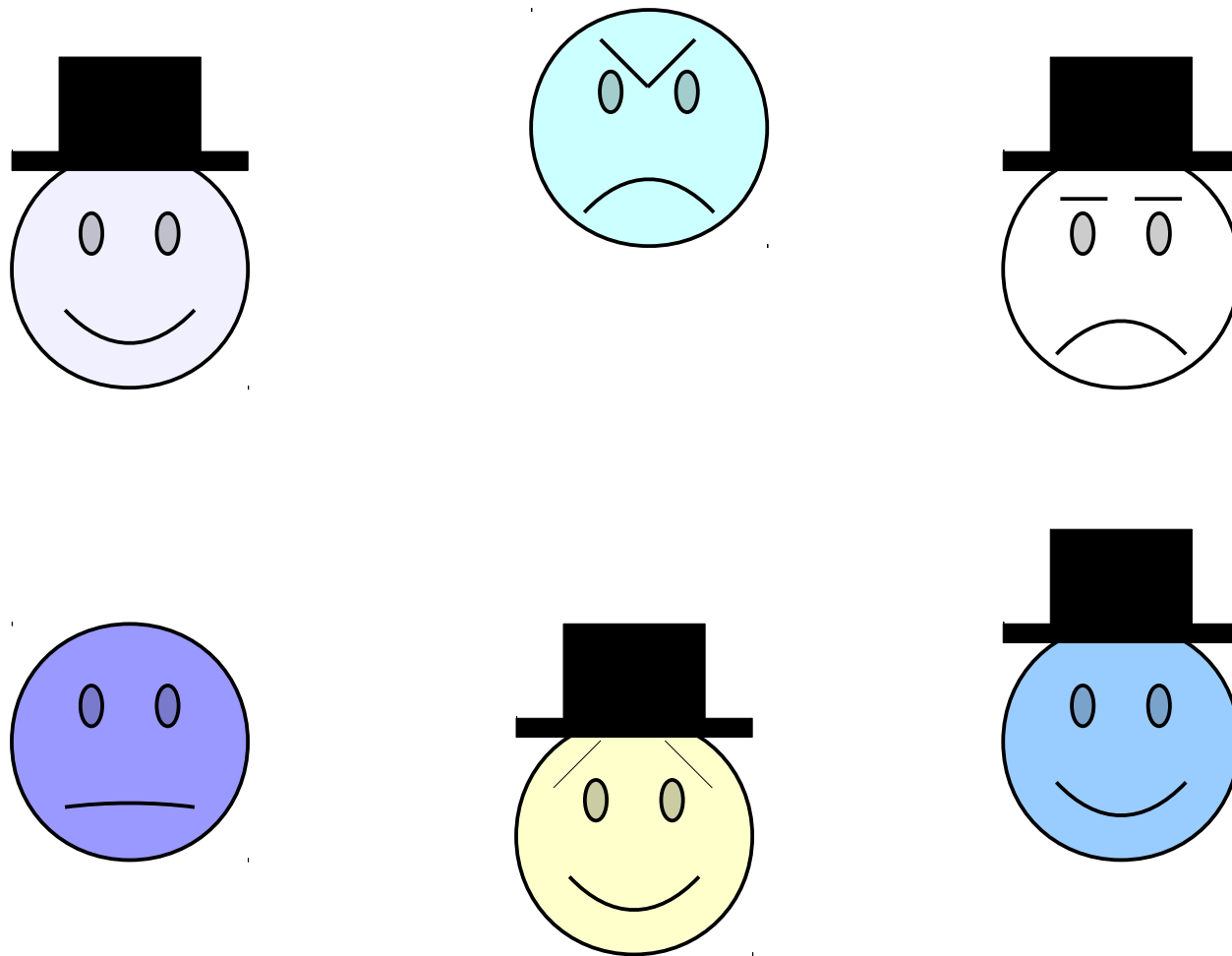
- *Smiling*(x), which states that x is smiling, and
- *WearingHat*(x), which states that x is wearing a hat,

write a sentence in first-order logic that says

every smiling person wears a hat.

Answer at

<https://pollev.com/cs103aut23>



“Every smiling person wears a hat.” **True**

~~$\forall x. (Smiling(x) \wedge WearingHat(x))$~~ **False**

$\forall x. (Smiling(x) \rightarrow WearingHat(x))$ **True**

“All P 's are Q 's”

translates as

$\forall x. (P(x) \rightarrow Q(x))$

Useful Intuition:

Universally-quantified statements are true unless there's a counterexample.

$$\forall x. (P(x) \rightarrow Q(x))$$

If x is a counterexample, it must have property P but not have property Q .

Good Pairings

- The \forall quantifier *usually* is paired with \rightarrow .

$$\forall x. (P(x) \rightarrow Q(x))$$

- The \exists quantifier *usually* is paired with \wedge .

$$\exists x. (P(x) \wedge Q(x))$$

- In the case of \forall , the \rightarrow connective prevents the statement from being *false* when speaking about some object you don't care about.
- In the case of \exists , the \wedge connective prevents the statement from being *true* when speaking about some object you don't care about.

Next Time

- ***First-Order Translations***
 - How do we translate from English into first-order logic?
- ***Quantifier Orderings***
 - How do you select the order of quantifiers in first-order logic formulas?
- ***Negating Formulas***
 - How do you mechanically determine the negation of a first-order formula?
- ***Expressing Uniqueness***
 - How do we say there's just one object of a certain type?